Practice - 12

Triple Integrals

8 701 273 47 85 – What's Up

Multiple integrals:

a) Double integrals

1) Iterated Integrals

2) Double Integrals over General Regions

3) Double Integrals in Polar Coordinates

b) Triple integrals

1) Triple Integrals in Cylindrical Coordinates

2) Triple Integrals in Spherical Coordinates

Triple Integrals

Now we want to integrate a function of three variables, $f(x,y,z)$.

Now that we know how to integrate over a two-dimensional region we need to move on to integrating over a three-dimensional region.

We used a double integral to integrate over a two-dimensional region and so it shouldn't be too surprising that we'll use a triple integral to integrate over a three dimensional region.

The notation for the general triple integrals is,

$$\iiint\limits_E f(x, y, z)\, dV$$

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$$\iiint\limits_E f(x,y,z)\ dV$$

Let's start simple by integrating over the box:

$$B = [a,b] \times [c,d] \times [r,s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is

$$\iiint\limits_B f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

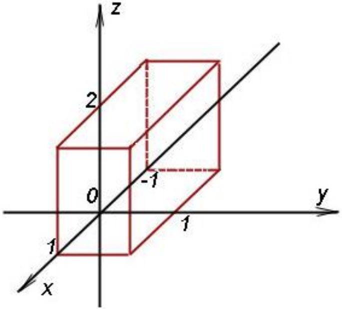
Ex1: Evaluate the following integral

$$\iiint\limits_B 8xyz\,dV \qquad B = [2,3] \times [1,2] \times [0,1]$$

Note that we integrated with respect to x first, then y, and finally z here, but in fact there is no reason to the integrals in this order.

Ex1: Evaluate the following integral

$$\iiint_B 8xyz\,dV \qquad B = [2,3] \times [1,2] \times [0,1]$$



$$\begin{aligned}
\iiint_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\
&= \int_1^2 \int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\
&= \int_1^2 \int_2^3 4xy\,dx\,dy \\
&= \int_1^2 2x^2y\Big|_2^3\,dy \\
&= \int_1^2 10y\,dy = 15
\end{aligned}$$

Task-1: Evaluate the following integral

$$I = \int_0^2 dx \int_{-1}^2 dy \int_1^e \frac{xy^3}{z} dz$$

Triple Integrals

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Step-1:

$$\int_1^e \frac{x^2 y^3}{z} dz = x^2 y^3 \ln |z| \Big|_1^e = x^2 y^3 (1-0) = x^2 y^3 .$$

Step-2:

$$\int_{-1}^2 x^2 y^3 dy = \frac{1}{4} x^2 y^4 \Big|_{-1}^2 = \frac{15}{4} x^2 .$$

Step-3:

$$\int_0^2 \frac{15}{4} x^2 dx = \frac{15}{4 \bullet 3} x^3 \Big|_0^2 = 10 .$$

Task-2: Evaluate the following integral

$$\iiint\limits_V (x+y+z)\,dx\,dy\,dz$$

V - parallelepiped bounded by planes:

$$x = -1,\ x = +1,\ y = 0,\ y = 1,\quad z = 0,\ z = 2.$$

Task-2: Evaluate the following integral $\iiint\limits_V (x+y+z)\,dx\,dy\,dz$,

$x=-1,\ x=+1,\ y=0,\ y=1,\ \ z=0,\ z=2.$

$$I=\int_{-1}^{1}dx\int_{0}^{1}dy\int_{0}^{2}(x+y-z)\,dz.$$

Step-1: $$\int_0^2 (x+y-z)\,dz=\left[xz+yz-\frac{z^2}{2}\right]\Bigg|_0^2=2x+2y-2.$$

Step-2: $$\int_0^2 (x+y-z)\,dz=\left[xz+yz-\frac{z^2}{2}\right]\Bigg|_0^2=2x+2y-2.$$

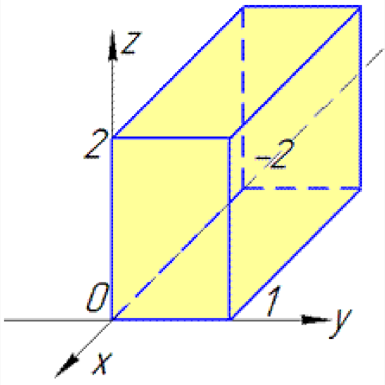
Step-3: $$\int_{-1}^{1}(2x-1)\,dx=\left[x^2-x\right]\Bigg|_{-1}^{1}=-2$$

Task-3: Evaluate the following integral

$$\iiint\limits_V (xy + 2z)\,dx\,dy\,dz$$

where

$$V: -2 \le x \le 0,\ 0 \le y \le 1,\ 0 \le z \le 2.$$



Triple Integrals

$$\iiint_V (xy+2z)\,dx\,dy\,dz = \int_{-2}^{0} dx \int_0^1 dy \int_0^2 (xy+2z)\,dz =$$

$$= \int_{-2}^{0} dx \int_0^1 \left(xy \cdot z + z^2 \right)\Big|_0^2 dy = \int_{-2}^{0} dx \int_0^1 (2xy+4)\,dy =$$

$$= \int_{-2}^{0} \left(x \cdot y^2 + 4 \cdot y \right)\Big|_0^1 dx = \int_{-2}^{0} (x+4)\,dx = \left(\frac{x^2}{2} + 4x \right)\Bigg|_{-2}^{0} = -2 + 8 = 6.$$

Task-33: Evaluate the following integral

$$\int_0^2 \int_0^z \int_0^y xyz\,dx\,dy\,dz.$$

Task-33: Evaluate the following integral

$$\int_0^2 \int_0^z \int_0^y xyz\,dx\,dy\,dz.$$

$$I = \int_0^2 \int_0^z \int_0^y xyz\,dx\,dy\,dz = \int_0^2 dz \int_0^z dy \int_0^y xyz\,dz = \int_0^2 dz \int_0^z dy \left[\left(\frac{x^2yz}{2} \right) \Big|_{x=0}^{x=y} \right] = \int_0^2 dz \int_0^z \frac{y^3 z}{2} dy$$

$$= \frac{1}{2} \int_0^2 dz \int_0^z y^3 z\,dy = \frac{1}{2} \int_0^2 dz \left[\left(\frac{y^4 z}{4} \right) \Big|_{y=0}^{y=z} \right] = \frac{1}{2} \int_0^2 \frac{z^5}{4} dz = \frac{1}{8} \int_0^2 z^5 dz = \frac{1}{8} \left(\frac{z^6}{6} \right) \Big|_0^2 = \frac{64}{48} = \frac{4}{3}.$$

Triple Integrals

The volume of the three-dimensional region E is given by the integral,

$$V = \iiint_E dV$$

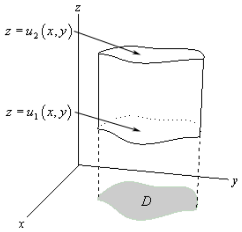
f(x,y,z)=1

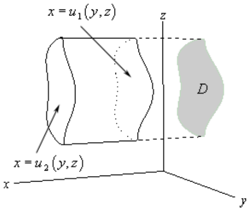
Triple Integrals Over General Regions

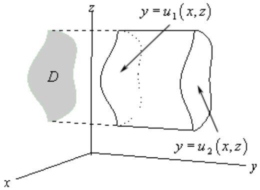
Let's now move on the more general three-dimensional regions.

We have three different possibilities for a general region.

A) B) C)





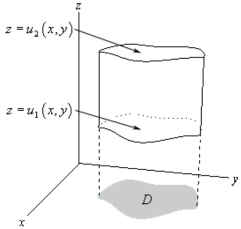


A) $$E = \{(x,y,z) \mid (x,y) \in D,\ \ u_1(x,y) \le z \le u_2(x,y)\}$$

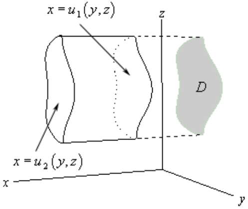
B) $$E = \{(x,y,z) \mid (y,z) \in D,\ \ u_1(y,z) \le x \le u_2(y,z)\}$$

C) $$E = \{(x,y,z) \mid (x,z) \in D,\ \ u_1(x,z) \le y \le u_2(x,z)\}$$

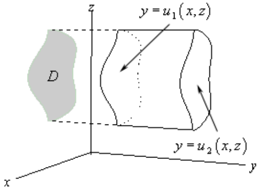
A)



B)



C)



A) $E = \{(x,y,z) \mid (x,y) \in D,\ \ u_1(x,y) \le z \le u_2(x,y)\}$

B) $E = \{(x,y,z) \mid (y,z) \in D,\ \ u_1(y,z) \le x \le u_2(y,z)\}$

C) $E = \{(x,y,z) \mid (x,z) \in D,\ \ u_1(x,z) \le y \le u_2(x,z)\}$

A) $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

B) $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

C) $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

Task-4: Evaluate the following integral

$$\iiint\limits_E 2x\, dV$$

where

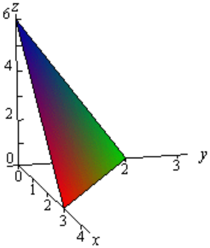
E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

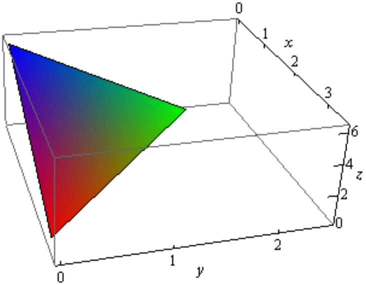
Task-4: Evaluate the following integral

$$\iiint_E 2x \, dV$$

where E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

We should first define octant. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.





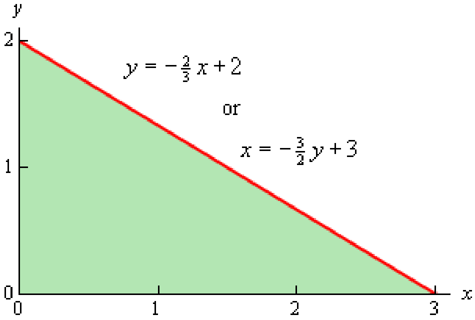
We now need to determine the region D in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region D in the xy-plane. So D will be the triangle with vertices at (0,0), (3,0), and (0,2).
Here is a sketch of D.

Step-1: $2x + 3y + z = 6$

*) $z = 6 - 2x - 3y$

**) $z = 0$

$2x + 3y = 6$



$y = -\frac{2}{3}x + 2$ or $x = -\frac{3}{2}y + 3$

Step-2: Now we need the limits of integration.

a) Since we are under the plane and in the first octant (so we're above the plane z=0) we have the following limits for z:

$$0 \leq z \leq 6 - 2x - 3y$$

We can integrate the double integral over D using either of the following two sets of inequalities.

b) $0 \leq x \leq 3$

c) $0 \leq y \leq -\frac{2}{3}x + 2$

or

bb) $0 \leq x \leq -\frac{3}{2}y + 3$

cc) $0 \leq y \leq 2$

Step-3: The integral is then

$$
\begin{aligned}
\iiint_E 2x\,dV &= \iint_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA \\
&= \iint_D 2xz\Big|_0^{6-2x-3y}\,dA \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\,(6-2x-3y)\;dy\,dx \\
&= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2}\,dx \\
&= \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx \\
&= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Big|_0^3 \\
&= 9
\end{aligned}
$$

Task-44:

$$\iiint\limits_U (1-x)\,dxdydz,$$

область U расположена в первом октанте ниже плоскости $3x+2y+z=6.$

$$3x + 2y + z = 6.$$

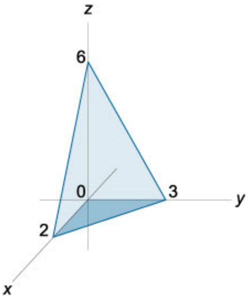
*) $z = 0$, $\quad 3x + 2y = 6.$

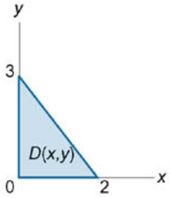
**) $z = 6 - 3x - 2y.$

$$3x + 2y = 6.$$

*) $y = 0$,

**) $y = 3 - \frac{3}{2}x$





$$I = \iiint\limits_U (1-x)\,dxdydz = \int_0^2 dx \int_0^{3-\frac{3}{2}x} dy \int_0^{6-3x-2y} (1-x)\,dz.$$

$$I = \int_0^2 dx \int_0^{3-\frac{3}{2}x} dy \int_0^{6-3x-2y} (1-x)\,dz = \int_0^2 dx \int_0^{3-\frac{3}{2}x} dy \left[(z - zx)\Big|_{z=0}^{z=6-3x-2y} \right]$$

$$= \int_0^2 dx \int_0^{3-\frac{3}{2}x} [6-3x-2y-(6-3x-2y)\,x]\,dy = \int_0^2 dx \int_0^{3-\frac{3}{2}x} \left(6-3x-2y-6x+3x^2+2xy\right) dy$$

$$= \int_0^2 dx \int_0^{3-\frac{3}{2}x} \left(6-9x-2y+3x^2+2xy\right) dy = \int_0^2 dx \left[\left(6y - 9xy - \frac{2y^2}{2} + 3x^2y + \frac{2xy^2}{2} \right)\Bigg|_{y=0}^{y=3-\frac{3}{2}x} \right]$$

$$= \int_0^2 \left(9 - 18x + \frac{45}{4}x^2 - \frac{9}{4}x^3 \right) dx = \left(9x - \frac{18}{2}x^2 + \frac{45}{12}x^3 - \frac{9}{16}x^4 \right)\Bigg|_0^2 = 18 - 36 + 30 - 9 = 3.$$

Task-5:

Determine the volume of the region that lies behind the plane $x + y + z = 8$

and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

Task-5:

Determine the volume of the region that lies behind the plane $x + y + z = 8$

and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

Step-1: $x + y + z = 8$

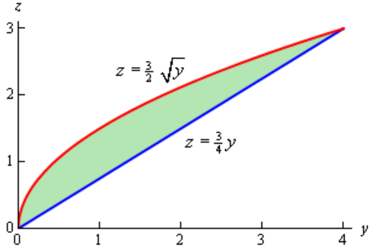
*) $x = 8 - y - z$

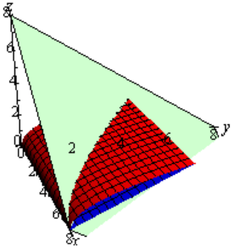
**) $x = 0$

$$0 \leq x \leq 8 - y - z$$

In this case we've been given D and so we won't have to really work to find that. Here is a sketch of the region D as well as a quick sketch of the plane and the curves defining D projected out past the plane so we can get an idea of what the region we're dealing with looks like.

Step-2:





$$0 \le y \le 4$$
$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

Step-3:

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_0^{8-y-z} dx\right] dA$$

$$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy$$

$$= \int_0^4 \left(8z - yz - \frac{1}{2}z^2\right)\Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy$$

$$= \int_0^4 12y^{\frac{1}{2}} - \frac{57}{8}y - \frac{3}{2}y^{\frac{3}{2}} + \frac{33}{32}y^2 \, dy$$

$$= \left(8y^{\frac{3}{2}} - \frac{57}{16}y^2 - \frac{3}{5}y^{\frac{5}{2}} + \frac{11}{32}y^3\right)\Bigg|_0^4 = \frac{49}{5}$$

Task-6:

$$\iiint\limits_V (x+y+z)\,dx\,dy\,dz$$

V is the pyramid bounded by the plane $x + y + z = 1$ and the coordinate planes $x = 0$, $y = 0$, $z = 0$.

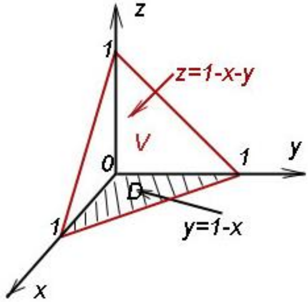
$$\iiint_V (x+y+z)\,dx\,dy\,dz\,,$$

V is the pyramid bounded by the plane x + y + z = 1 and the coordinate planes x = 0, y = 0, z = 0.

Step-1: x + y + z = 1

*) z=0: x+y=1

**) z=1-x -y



Step-2: x+y=1: y=1-x

x=0: y=1

y=0: x=1

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z)\,dz\,.$$

Triple Integrals

$$I=\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}(x+y+z)\,dz.$$

1) $$\int_0^{1-x-y}(x+y+z)\,dz=\left[xz+yz+\frac{z^2}{2}\right]\Bigg|_0^{1-x-y}.$$

2) $$\int_0^{1-x}\left[xz+yz+\frac{z^2}{2}\right]\Bigg|_0^{1-x-y}dy=\frac{1}{2}\left[y-yx^2-xy^2-\frac{y^3}{3}\right]\Bigg|_0^{1-x}.$$

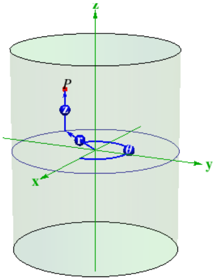
3) $$\frac{1}{2}\int_0^1\left[y-yx^2-xy^2-\frac{y^3}{3}\right]\Bigg|_0^{1-x}dx=\frac{1}{6}\int_0^1\left(2-3x+x^3\right)dx=$$

$$=\frac{1}{6}\left[2x-\frac{3x^2}{2}+\frac{x^4}{4}\right]\Bigg|_0^1=\frac{1}{6}\bullet\frac{3}{4}=\frac{1}{8}.$$

Triple Integrals In Cylindrical Coordinates

Recall that cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$



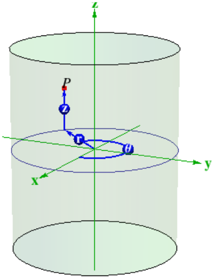
$$\iiint_E f(x,y,z)\,dV$$

In order to do the integral in cylindrical coordinates we will need to know what dV will become in terms of cylindrical coordinates.

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \frac{\partial x}{\partial \theta} = -r\sin(\theta), \ \frac{\partial x}{\partial z} = 0,$$
$$\frac{\partial y}{\partial r} = \sin(\theta), \ \frac{\partial y}{\partial \theta} = r\cos(\theta), \ \frac{\partial y}{\partial z} = 0,$$
$$\frac{\partial z}{\partial r} = 0, \qquad \frac{\partial z}{\partial \theta} = 0, \qquad \frac{\partial z}{\partial z} = 1.$$



Our Jacobian is then the 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

$$dV = r\,dz\,dr\,d\theta$$

and our volume element is
$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz.$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\iiint\limits_E f(x,y,z)\,dV$$

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

The region E, over which we are integrating becomes

$$\begin{aligned} E &= \{(x,y,z)\,|\,(x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\} \\ &= \{(r,\theta,z)\,|\alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta)\le z\le u_2(r\cos\theta, r\sin\theta)\} \end{aligned}$$

In terms of cylindrical coordinates a triple integral is

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

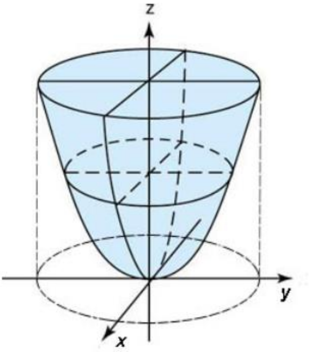
Ex2: $$\iiint_V \left(x^2+y^2\right) dx\, dy\, dz$$

V is the domain bounded by $z = x^2 + y^2$ and $z = 1$.

$$x^2 + y^2 \le 1$$

*) $x^2 + y^2 = r^2$. $\quad z = x^2 + y^2 = r^2$.

**) $z = 1$.



$$\iiint_V \left(x^2+y^2\right) dx\, dy\, dz =$$

$$= \int_0^{2\pi} d\varphi \int_0^1 dr \int_{r^2}^1 r^2 \bullet r\, dz =$$

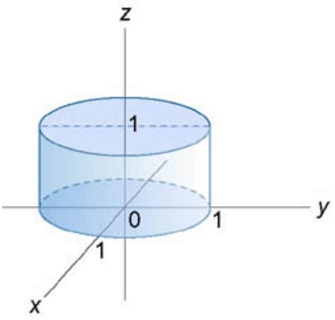
$$= \int_0^{2\pi} d\varphi \int_0^1 \left[r^2 z\right]\Big|_{r^2}^1 dr =$$

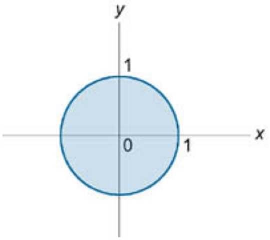
$$= \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^6}{6}\right]\Bigg|_0^1 d\varphi =$$

$$= \frac{1}{12}\int_0^{2\pi} d\varphi = \frac{1}{12}\varphi\Big|_0^{2\pi} = \frac{\pi}{6}.$$

Ex22: $$\iiint\limits_U \left(x^4 + 2x^2y^2 + y^4\right) dxdydz,$$

U is the domain bounded by $x^2 + y^2 \leq 1$ and $z = 0, z = 1$





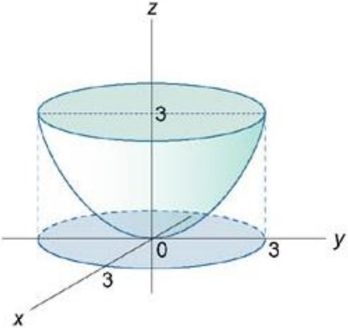
Triple Integrals In Cylindrical Coordinates

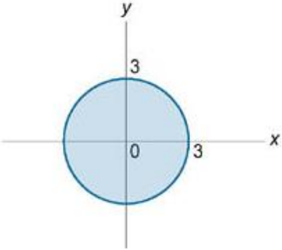
$$\left(x^4 + 2x^2y^2 + y^4\right) = \left(x^2 + y^2\right)^2 = \left(\rho^2\right)^2 = \rho^4.$$

$$I = \int_0^{2\pi} d\varphi \int_0^1 \rho^4 \rho d\rho \int_0^1 dz = 2\pi \int_0^1 \rho^5 d\rho \int_0^1 dz = 2\pi \cdot 1 \cdot \int_0^1 \rho^5 d\rho = 2\pi \left(\frac{\rho^6}{6} \right) \Big|_0^1 = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}.$$

Ex222: $$\iiint\limits_U (x^2+y^2)\,dxdydz,$$

U is the domain bounded by $x^2+y^2=3z,\ z=3$





$$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi = 3z \;\text{ или }\; \rho^2 = 3z.$$

Проекция области интегрирования U на плоскость Oxy представляет собой окружность $x^2 + y^2 \leq 9$ радиусом $\rho = 3$ (рисунок 5). Координата ρ изменяется в пределах от 0 до 3, угол φ – от 0 до 2π и координата z – от $\frac{\rho^2}{3}$ до 3. В результате интеграл будет равен

$$I = \iiint\limits_U (x^2+y^2)\,dxdydz = \iiint\limits_{U'} \rho^2 \cdot \rho d\rho d\varphi dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \rho^3 d\rho \int\limits_{\frac{\rho^2}{3}}^3 dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \rho^3 d\rho \cdot \left[z\Big|_{\frac{\rho^2}{3}}^3 \right]$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \rho^3\left(3 - \frac{\rho^2}{3}\right) d\rho = \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \left(3\rho^3 - \frac{\rho^5}{3}\right) d\rho = \int\limits_0^{2\pi} d\varphi \cdot \left[\left(\frac{3\rho^4}{4} - \frac{\rho^6}{18}\right)\Bigg|_0^3 \right]$$

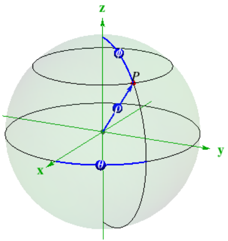
$$= \left(\frac{3\cdot 81}{4} - \frac{729}{18}\right)\int\limits_0^{2\pi} d\varphi = \frac{81}{4}\cdot 2\pi = \frac{81\pi}{2}.$$

We looked at doing integrals in terms of cylindrical coordinates and we now need to take a quick look at doing integrals in terms of spherical coordinates.

Here are the conversion formulas for spherical coordinates:

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi.$$

$$0 \le \rho < \infty, \qquad 0 \le \theta < 2\pi, \qquad 0 \le \phi \le \pi.$$



Triple Integrals In Spherical Coordinates

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi.$$

$$\frac{\partial x}{\partial \rho} = \cos(\theta)\sin(\phi), \frac{\partial x}{\partial \theta} = -\rho\sin(\theta)\sin(\phi), \ \frac{\partial x}{\partial \phi} = \rho\cos(\theta)\cos(\phi),$$
$$\frac{\partial y}{\partial \rho} = \sin(\theta)\sin(\phi), \ \frac{\partial y}{\partial \theta} = \rho\cos(\theta)\sin(\phi), \ \frac{\partial y}{\partial \phi} = \rho\sin(\theta)\cos(\phi),$$
$$\frac{\partial z}{\partial \rho} = \cos(\phi), \qquad \frac{\partial z}{\partial \theta} = 0, \qquad \frac{\partial z}{\partial \phi} = -\rho\sin(\phi).$$

Our Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ is then the 3×3 determinant

$$\begin{vmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi). \end{vmatrix}$$

which works out to $\rho^2\sin(\phi)$, and our volume element is $dV = dx\,dy\,dz = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$.

$$\boxed{dV = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi}$$

Triple Integrals In Spherical Coordinates

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi.$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$\iiint\limits_E f(x, y, z)\, dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b \rho^2 \sin\varphi \; f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\; d\rho\, d\theta\, d\varphi$$

Task-7:

$$\text{Evaluate} \iiint_E 16z\,dV$$

where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Evaluate $\displaystyle\iiint\limits_E 16z\,dV$

where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi.$$

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

Triple Integrals In Spherical Coordinates

$$
\begin{aligned}
\iiint\limits_E 16z\,dV &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi\,(16\rho\cos\varphi)\;d\rho\,d\theta\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 8\rho^3 \sin(2\varphi)\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin(2\varphi)\,d\theta\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} 4\pi\sin(2\varphi)\,d\varphi \\
&= -2\pi\cos(2\varphi)\big|_0^{\frac{\pi}{2}} \\
&= 4\pi
\end{aligned}
$$

Task-8:

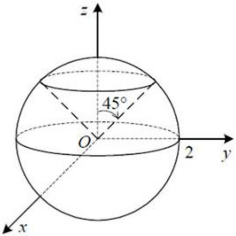
Calculate the integral
$$\iiint\limits_V z\sqrt{x^2+y^2}\,dxdydz$$

with the transition to spherical coordinates, where V is the region bounded by the inequalities:
$$z \geq \sqrt{x^2+y^2}$$
$$x^2+y^2+z^2 \leq 4.$$

$$z \geq \sqrt{x^2 + y^2}$$

$$x^2 + y^2 + z^2 \leq 4.$$

$$0 \leq \rho \leq 2, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \frac{\pi}{4}.$$



$$z\sqrt{x^2 + y^2} =$$

$$= \rho \cos\theta \sqrt{(\rho \sin\theta \cos\varphi)^2 + (\rho \sin\theta \sin\varphi)^2} =$$

$$= \rho \cos\theta \sqrt{\rho^2 \sin^2\theta \left(\cos^2\varphi + \sin^2\varphi\right)} =$$

$$= \rho \cos\theta \bullet \rho \sin\theta = \rho^2 \sin\theta \cos\theta.$$

$$\iiint_V z\sqrt{x^2+y^2}\,dxdydz =$$

$$= \iiint_\Omega \rho^2 \sin\theta \cos\theta \bullet \rho^2 \sin\theta d\rho d\varphi d\theta =$$

$$= \iiint_\Omega \rho^4 \sin^2\theta \cos\theta d\rho d\varphi d\theta.$$

Triple Integrals In Spherical Coordinates

$$\iiint\limits_{\Omega} \rho^4 \sin^2\theta \cos\theta d\rho d\varphi d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta \cos\theta d\theta \int_0^{2\pi} d\varphi \int_0^{2} \rho^4 d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta d\left(\sin\theta\right) \bullet \varphi\Big|_0^{2\pi} \bullet \frac{\rho^5}{5}\Bigg|_0^{2} =$$

$$= 2\pi \bullet \frac{32}{5} \bullet \frac{\sin^3\theta}{3}\Bigg|_0^{\frac{\pi}{4}} =$$

$$= \frac{64\pi}{15}\left(\sin^3\frac{\pi}{4} - \sin^3 0\right) =$$

$$= \frac{64\pi}{15}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - 0\right) =$$

$$= \frac{64\pi}{15} \bullet \frac{\sqrt{2}}{4} = \frac{16\pi\sqrt{2}}{15}.$$

Homework

Рябушко, часть 3.

АЗ-13.5

1. Вычислить $\iiint\limits_V x^2y^2z\,dxdydz$, если область V определяется неравенствами $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant x$, $0 \leqslant z \leqslant xy$. (*Ответ:* $1/110$.)

2. Вычислить $\iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3}$, если область V ограничена плоскостями $x=0$, $y=0$, $z=0$, $x+y+z=1$. $\left(\textit{Ответ: } \frac{1}{2}\left(\ln 2 - \frac{5}{8}\right).\right)$

3. Вычислить объем тела, ограниченного поверхностями $y=x^2$, $y+z=4$, $z=0$. (*Ответ:* $256/15$.)

Рябушко, часть 3.

Самостоятельная работа

1. 1. Расставить пределы интегрирования в интеграле $\iiint\limits_V f(x, y, z)dxdydz$, если область V ограничена плоскостями $x=0$, $y=0$, $z=0$, $2x+3y+4z=12$.

2. Вычислить $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$, если область V ограничена поверхностями $z=x^2+y^2$, $z=1$. (*Ответ:* $4\pi/15$.)

2. 1. Расставить пределы интегрирования в интеграле $\iiint\limits_V f(x, y, z)dxdydz$, если область V ограничена поверхностями $y=x$, $y=2x$, $z=0$, $x+z=2$.

2. Вычислить $\iiint\limits_V \sqrt{x^2+z^2}\,dxdydz$, если область V ограничена поверхностями $y=x^2+z^2$, $z=1$. (*Ответ:* $4\pi/15$.)

Рябушко, часть 3.

ИДЗ-13.2

Решения всех вариантов тут >>>

1. Расставить пределы интегрирования в тройном интеграле $\iiint\limits_V f(x, y, z)dxdydz$, если область V ограничена указанными поверхностями. Начертить область интегрирования.

1.1. V: $x = 2$, $y = 4x$, $y = 3\sqrt{x}$; $z \geqslant 0$, $z = 4$.
1.2. V: $x = 1$; $y = 3x$, $y \geqslant 0$, $z \geqslant 0$, $z = 2(x^2 + y^2)$.
1.3. V: $x = 1$, $y = 4x$, $z \geqslant 0$, $z = \sqrt{3y}$.
1.4. V: $x = 3$, $y = x$, $y \geqslant 0$, $z \geqslant 0$, $z = 3x^2 + y^2$.
1.5. V: $y = 2x$, $y = 2$, $z \geqslant 0$, $z = 2\sqrt{x}$.

Рябушко, часть 3.

2. Вычислить данные тройные интегралы.

2.1. $\iiint\limits_V (2x^2+3y+z)dxdydz$, V: $2 \leqslant x \leqslant 3$, $-1 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 4$.

2.2. $\iiint\limits_V x^2yzdxdydz$, V: $-1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 3$, $2 \leqslant z \leqslant 3$.

2.3. $\iiint\limits_V (x+y+4z^2)dxdydz$, V: $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $-1 \leqslant z \leqslant 1$.

2.4. $\iiint\limits_V (x^2+y^2+z^2)dxdydz$; V: $0 \leqslant x \leqslant 3$, $-1 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 2$.

2.5. $\iiint\limits_V x^2y^2zdxdydz$, V: $-1 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 2$, $-2 \leqslant z \leqslant 5$.